

Semi-Classical Approach to the Two-Level Pairing Model

— *Various Aspects of Phase Change* —

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Abstract

Aspects of the phase change of the two-level pairing model are investigated in the semi-classical treatment by using the variational approach with the mixed-mode coherent state. In the classical limit, $\hbar \rightarrow 0$, the sharp phase transition appears and the two phases exist in the region where the force strength is larger than a certain critical value. However, it is shown that, in the semi-classical treatment, the above-mentioned behavior of the phase change disappears in the analytical and the numerical treatments. As a result, new understanding about aspects of the phase change in this model is given.

§1. Introduction

The phase structure and the phase transitions between various phases in condensed matter systems, such as nuclear and/or quark matter as well as the condensed matter in atoms and molecules, are important concepts to understand the nature of the matter under consideration. In the infinite systems, the clear phase separation is often realized and, in that case, the sharp phase transition may occur among the different phases. On the other hand, in finite systems such as nucleus, the phase change is not so clear. This situation can be demonstrated¹⁾ in the exactly-solvable many-particle models such as the Lipkin model²⁾ or the pairing model.³⁾ However, the mean field approximation in the time-independent theory, which often leads to the classical treatment, gives the fictitious sharp phase transition. Addition to the above mentioned problem, the two phases, namely, the condensed and the non-condensed phases, appear in both the Lipkin and the pairing models in the classical treatment, if the force strength is larger than a certain critical value at zero temperature.

In this paper, an exactly-solvable two-level pairing model is treated to understand the nature of the phase change and the existence of two phases for the force strength at zero temperature in the classical limit. The two-level pairing model is regarded as the $su(2) \otimes su(2)$ -algebraic model. For the original many-fermion system, the trial state for the variation is restricted to the BCS state which gives the mean field approximation and the unique classical counterpart for the original quantal system. In order to avoid the usual mean field approximation of the BCS theory, the two-level pairing model can be expressed in the four kinds of boson operators by using the Schwinger boson representation of the $su(2)$ -algebra. In this boson representation, the various trial states are possible to obtain a classical counterpart.⁴⁾ Thus, in this paper, a certain wave packet which we call the mixed-mode coherent state⁵⁾ is adopted to the variational calculation. This state gives the semi-classical approximation which retains \hbar

This paper is organized as follows: In the next section, the two-level pairing model is introduced by means of the Schwinger boson representation. In §3, the classical counterpart for the two-level pairing model is given by using the mixed-mode coherent state and the basic equations of motion in this approach are also derived. In §4, the energy minimum point is investigated in the classical limit $\hbar \rightarrow 0$ and in the semi-classical approach with finite value of \hbar . Then, it is shown that the sharp phase transition does not occur with finite \hbar , while the sharp phase transition occurs and two phases exist in the certain region in the classical limit. The method to obtain an approximate solution for the energy minimum is given analytically in §5 together with appendix A. The behavior of the phase change is explained with unknown concept in §6 and the discussion about the ground state energy is

given in §7. The last section is devoted to a summary.

§2. Two-level pairing model in many-fermion system and its counterpart in the Schwinger boson representation

The model discussed in this paper is two-level pairing model in many-fermion system. The two levels are specified by $\sigma = +$ (the upper) and $\sigma = -$ (the lower), respectively. The difference of the single-particle energies between the two levels and the strength of the pairing interaction are denoted as $\hbar\epsilon$ (> 0) and $\hbar^2 G$ (> 0), respectively. The Hamiltonian, \tilde{H} , is expressed in the form

$$\tilde{H} = \epsilon(\tilde{S}_0(+) - \tilde{S}_0(-)) - G(\tilde{S}_+(+) + \tilde{S}_+(-))(\tilde{S}_-(+) + \tilde{S}_-(-)) . \quad (2.1)$$

Here, $\tilde{S}_0(\sigma)$, $\tilde{S}_+(\sigma)$ and $\tilde{S}_-(\sigma)$ are defined as

$$\tilde{S}_0(\sigma) = (\hbar/2)(\tilde{N}(\sigma) - \Omega_\sigma) , \quad (2.2a)$$

$$\tilde{S}_+(\sigma) = (\hbar/2)\tilde{P}^*(\sigma) , \quad \tilde{S}_-(\sigma) = (\hbar/2)\tilde{P}(\sigma) , \quad (2.2b)$$

$$\tilde{N}(\sigma) = \sum_{m=-j_\sigma}^{j_\sigma} \tilde{c}_m^*(\sigma)\tilde{c}_m(\sigma) , \quad \Omega(\sigma) = (2j_\sigma + 1)/2 , \quad (2.3a)$$

$$\tilde{P}^*(\sigma) = \sum_{m=-j_\sigma}^{j_\sigma} \tilde{c}_m^*(\sigma)(-)^{j_\sigma-m}\tilde{c}_{-m}^*(\sigma) , \quad \tilde{P}(\sigma) = \sum_{m=-j_\sigma}^{j_\sigma} (-)^{j_\sigma-m}\tilde{c}_{-m}(\sigma)\tilde{c}_m(\sigma) . \quad (2.3b)$$

The operator $(\tilde{c}_m^*(\sigma), \tilde{c}_m(\sigma))$ denotes fermion operator in the single-particle state $(\sigma, j_\sigma, m; m = -j_\sigma, -j_\sigma + 1, \dots, j_\sigma - 1, j_\sigma)$. Of course, j_σ is a half-integer. Clearly, $\tilde{N}(\sigma)$ and $(\tilde{P}^*(\sigma), \tilde{P}(\sigma))$ denote the fermion number and the fermion pair operators in the state σ , respectively.

We know that the set $(\tilde{S}_{\pm,0}(\sigma))$ obeys the $su(2)$ -algebra, and then, essentially, the presented model is governed by the $su(2) \otimes su(2)$ -algebra. Clearly, the addition of the two sets also obeys the $su(2)$ -algebra:

$$\tilde{S}_{\pm,0} = \tilde{S}_{\pm,0}(+) + \tilde{S}_{\pm,0}(-) . \quad (2.4)$$

The Hamiltonian (2.1) has three constants of motion. This can be shown from the following relation:

$$[\tilde{\mathbf{S}}^2(+), \tilde{H}] = [\tilde{\mathbf{S}}^2(-), \tilde{H}] = [\tilde{S}_0, \tilde{H}] = 0 . \quad (2.5)$$

Here, $\tilde{\mathbf{S}}^2(\sigma)$ denotes the Casimir operator for $(\tilde{S}_{\pm,0}(\sigma))$. It should be noted that the Casimir operators for $(\tilde{S}_{\pm,0})$, $\tilde{\mathbf{S}}^2$, does not commute with \tilde{H} :

$$[\tilde{\mathbf{S}}^2, \tilde{H}] \neq 0 . \quad (2.6)$$

A general framework of the $su(2) \otimes su(2)$ -algebra can be formulated in a form of the Schwinger boson representation, in which four kinds of boson operators are used. A possible example of this formulation was performed by the present authors (J.P. and M.Y.) with Kuriyama.⁵⁾ Hereafter, we will refer it as (A). In (A), the $su(2) \otimes su(2)$ -algebra is formulated in terms of the bosons $(\hat{a}_\sigma^*, \hat{a}_\sigma, \hat{b}_\sigma^*, \hat{b}_\sigma; \sigma = \pm)$. In this framework, we can construct a counterpart of the model presented in this section in the Schwinger boson representation. The $su(2)$ -generators $(\tilde{S}_{\pm,0}(\sigma); \sigma = \pm)$ correspond to $\hat{S}_{\pm,0}(\sigma)$ in the form

$$\tilde{S}_{\pm,0}(\sigma) \longrightarrow \hat{S}_{\pm,0}(\sigma) , \quad (2.7)$$

$$\begin{aligned} \hat{S}_0(\sigma) &= (\hbar/2)(\hat{a}_\sigma^* \hat{a}_\sigma - \hat{b}_\sigma^* \hat{b}_\sigma) , \\ \hat{S}_+(\sigma) &= \hbar \hat{a}_\sigma^* \hat{b}_\sigma , \quad \hat{S}_-(\sigma) = \hbar \hat{b}_\sigma^* \hat{a}_\sigma . \end{aligned} \quad (2.8)$$

It should be noted that the notations are different from those in (A). Of course, we have

$$\begin{aligned} \hat{S}_0 &= (\hbar/2)(\hat{a}_+^* \hat{a}_+ - \hat{b}_+^* \hat{b}_+ + \hat{a}_-^* \hat{a}_- - \hat{b}_-^* \hat{b}_-) , \\ \hat{S}_+ &= \hbar(\hat{a}_+^* \hat{b}_+ + \hat{a}_-^* \hat{b}_-) , \\ \hat{S}_- &= \hbar(\hat{b}_+^* \hat{a}_+ + \hat{b}_-^* \hat{a}_-) . \end{aligned} \quad (2.9)$$

Associating with the above operators, we can define the following operators:

$$\hat{S}(\sigma) = (\hbar/2)(\hat{a}_\sigma^* \hat{a}_\sigma + \hat{b}_\sigma^* \hat{b}_\sigma) . \quad (2.10)$$

This operator satisfies, for the Casimir operator $\hat{\mathbf{S}}^2(\sigma)$, the relation

$$\hat{\mathbf{S}}^2(\sigma) = \hat{S}(\sigma)(\hat{S}(\sigma) + \hbar) . \quad (2.11)$$

In the case where the seniorities are equal to 0 in the fermion system, we should set up the correspondence

$$(\hbar/2)\Omega(\sigma) \longrightarrow \hat{S}(\sigma) . \quad (2.12)$$

In the Schwinger boson representation, there does not exist the concept of the fermion number explicitly. The definition of $\tilde{S}_0(\sigma)$ given in the relation (2.2a) and the correspondence (2.12) permit us to introduce the fermion number in the Schwinger boson representation in the form

$$\hbar \hat{N}(\sigma) = 2(\hat{S}(\sigma) + \hat{S}_0(\sigma)) = 2\hbar \hat{a}_\sigma^* \hat{a}_\sigma . \quad (2.13)$$

Therefore, the total number \hat{N} is given as

$$\begin{aligned} \hbar \hat{N} &= \hbar(\hat{N}(+) + \hat{N}(-)) = 2(\hat{S}(+) + \hat{S}(-) + \hat{S}_0) \\ &= 2\hbar(\hat{a}_+^* \hat{a}_+ + \hat{a}_-^* \hat{a}_-) . \end{aligned} \quad (2.14)$$

From the above form, we can see that in the present case for both levels, even fermion numbers should be occupied. In the case of odd fermion numbers, we must take into account the case where the seniorities do not vanish.

§3. Classical counterpart of the $su(2) \otimes su(2)$ -algebra

We continue the recapitulation of (A). In (A), we showed that the orthogonal set for the $su(2) \otimes su(2)$ -algebra is obtained with the use of the set $(\hat{R}_{\pm,0}, \hat{T}_{\pm,0})$ as a supporting role. The set is defined in the form

$$\begin{aligned} \hat{R}_0 &= (\hbar/2)(\hat{a}_+^* \hat{a}_+ - \hat{a}_-^* \hat{a}_-) , \\ \hat{R}_+ &= \hbar(\hat{a}_+^* \hat{a}_- + \hat{b}_+^* \hat{b}_-) , \quad \hat{R}_- = \hbar(\hat{a}_-^* \hat{a}_+ + \hat{b}_-^* \hat{b}_+) , \end{aligned} \quad (3.1)$$

$$\begin{aligned} \hat{T}_0 &= (\hbar/2)(\hat{a}_+^* \hat{a}_+ + \hat{b}_+^* \hat{b}_- + \hat{a}_-^* \hat{a}_- + \hat{b}_+^* \hat{b}_+ + 2) , \\ \hat{T}_+ &= \hbar(\hat{a}_+^* \hat{b}_-^* - \hat{a}_-^* \hat{b}_+^*) , \quad \hat{T}_- = \hbar(\hat{b}_- \hat{a}_+ - \hat{b}_+ \hat{a}_-) , \end{aligned} \quad (3.2)$$

The sets $(\hat{R}_{\pm,0})$ and $(\hat{T}_{\pm,0})$ obey the $su(2)$ - and the $su(1,1)$ -algebra, respectively, and it is characteristic that three sets $(\hat{S}_{\pm,0})$, $(\hat{R}_{\pm,0})$ and $(\hat{T}_{\pm,0})$ commute with one another. Further, the Casimir operators for the three forms are identically equal to one another.

In (A), we defined a wave packet, which we called a mixed-mode coherent state, as follows:

$$\begin{aligned} |c\rangle &= U_t^{-2} \exp(-|W|^2/\hbar) \cdot \exp\left((V_t/\hbar U_t)\hat{T}_+\right) \cdot \exp\left((V_s/\hbar U_s)\hat{S}_+\right) \\ &\times \exp\left((V_r/\hbar U_r)\hat{R}_+\right) \cdot \exp\left(\left(\sqrt{\hbar/2} W U_r U_s / U_t\right) \hat{b}_-^*\right) |0\rangle . \end{aligned} \quad (3.3)$$

The state $|c\rangle$ is normalized and U_r , U_s and U_t are real and W , V_r , V_s and V_t complex. They obey the condition

$$U_t^2 - |V_t|^2 = 1 , \quad U_s^2 + |V_s|^2 = 1 , \quad U_r^2 + |V_r|^2 = 1 . \quad (3.4)$$

We can prove the following relation:

$$\begin{aligned} \hat{a}_+ &= U_t \hat{A}_+ + V_t \hat{B}_-^* + \sqrt{\frac{2}{\hbar}}(W V_r V_s U_t + W^* U_r U_s V_t) , \\ \hat{a}_- &= U_t \hat{A}_- - V_t \hat{B}_+^* + \sqrt{\frac{2}{\hbar}}(W U_r V_s U_t - W^* V_r^* U_s V_t) , \\ \hat{b}_+ &= U_t \hat{B}_+ - V_t \hat{A}_-^* + \sqrt{\frac{2}{\hbar}}(W V_r U_s U_t - W^* U_r V_s^* V_t) , \\ \hat{b}_- &= U_t \hat{B}_- + V_t \hat{A}_+^* + \sqrt{\frac{2}{\hbar}}(W U_r U_s U_t + W^* V_r^* V_s^* V_t) . \end{aligned} \quad (3.5)$$

Here, $(\hat{A}_\sigma, \hat{A}_\sigma^*)$ and $(\hat{B}_\sigma, \hat{B}_\sigma^*)$ denote boson operators satisfying

$$\hat{A}_\sigma |c\rangle = 0 , \quad \hat{B}_\sigma |c\rangle = 0 . \quad (3.6)$$

As a possible parameterization, in (A), the following form were presented:

$$\begin{aligned}
W &= \sqrt{S} e^{-i(\psi - \psi_0 + \psi(-))/2} , \\
V_r &= \sqrt{(S + S(+)) - S(-))/2S} e^{-i(\psi(+) - \psi(-))/2} , \\
V_s &= \sqrt{(S + S_0)/2S} e^{-i\psi_0} , \\
V_t &= \sqrt{(S(+) + S(-) - S)/2(S + \hbar)} e^{-i(\psi(+) + \psi(-))/2} , \\
U_r &= \sqrt{(S - S(+)) + S(-))/2S} , \quad U_s = \sqrt{(S - S_0)/2S} , \\
U_t &= \sqrt{(S(+) + S(-) + S + 2\hbar)/2(S + \hbar)} .
\end{aligned} \tag{3.7}$$

In (A), we prove that $(\psi(\sigma), S(\sigma))$, (ψ_0, S_0) and (ψ, S) are in the relations of the angle and the action variables in classical mechanics. Further, we have the following relation:

$$S \geq 0 , \quad -S \leq S_0 \leq S , \quad |S(+) - S(-)| \leq S \leq S(+) + S(-) . \tag{3.8}$$

The above is nothing but the relation that implies the coupling of two $su(2)$ -spins.

With the aid of the relation (3.5), we can calculate the expectation value of any operator composed of the present bosons with respect to $|c\rangle$: We denote $\langle c|\hat{O}|c\rangle = (\hat{O})_c$. For example, we have

$$(\hat{S}(\sigma))_c = S(\sigma) , \tag{3.9}$$

$$(\hat{S}_0)_c = S_0 ,$$

$$(\hat{S}_\pm)_c = \sqrt{S^2 - S_0^2} e^{\pm i\psi_0} . \tag{3.10}$$

The relation (3.10) is identical with the classical counterpart of the Holstein-Primakoff boson representation, and then, including the relation (3.8), our present formalism may be called the classical counterpart of the $su(2) \otimes su(2)$ -algebra. Concerning $\hat{S}_+ \hat{S}_-$, the most interesting point is as following:

$$(\hat{S}_+ \hat{S}_-)_c = ((\hat{S}_+(+) + \hat{S}_+(-))(\hat{S}_- (+) + \hat{S}_- (-)))_c = S^2 - S_0^2 + \hbar(S + S_0) . \tag{3.11}$$

$$(\hat{S}_+)_c (\hat{S}_-)_c = S^2 - S_0^2 . \tag{3.12}$$

We can see that in the relation (3.11), the quantal fluctuation is exactly taken into account. Further, we have

$$\begin{aligned}
& (\hat{S}_0(+) - \hat{S}_0(-))_c \\
&= \frac{(S(+) - S(-))(S(+) + S(-) + \hbar)S_0}{S(S + \hbar)} \\
&+ \frac{\sqrt{(S - S(+) + S(-))(S + S(+) - S(-))(S(+) + S(-) - S)(S(+) + S(-) + S + 2\hbar)}}{S(S + \hbar)} \cos \psi .
\end{aligned} \tag{3.13}$$

Under the above preparation, we are able to obtain the classical Hamiltonian of \hat{H} which is the counterpart of \tilde{H} shown in the relation (2.1):

$$\hat{H} = \epsilon(\hat{S}_0(+) - \hat{S}_0(-)) - G(\hat{S}_+(+) + \hat{S}_+(-))(\hat{S}_-(+) + \hat{S}_-(-)) . \quad (3.14)$$

The Hamiltonian \hat{H} obeys the condition

$$[\hat{S}(+) , \hat{H}] = [\hat{S}(-) , \hat{H}] = [\hat{S}_0 , \hat{H}] = 0 . \quad (3.15)$$

The relation (3.15) corresponds to the relation (2.5). It is enough to calculate the expectation value of \hat{H} for $|c\rangle$: $(\hat{H})_c = H$. In this case, the forms (3.11) and (3.13) are useful. Of course, the classical expression of the fermion number is given in the form

$$\hbar N = \hbar(\hat{N})_c = 2(S(+) + S(-) + S_0) . \quad (3.16)$$

For the above, the relation (2.14) is used. Corresponding to the Hamiltonian \hat{H} , H satisfies

$$[S(+) , H]_P = [S(-) , H]_P = [S_0 , H]_P = 0 . \quad (3.17)$$

Here, $[,]_P$ denotes the Poisson bracket. As was already mentioned, the parameters $(\psi(\sigma), S(\sigma))$, (ψ_0, S_0) and (ψ, S) are in the relation of the canonical variables in classical mechanics. From the relation (3.17), we can see that $S(+)$, $S(-)$ and S_0 are constants of motion and the time-evolution of the angles $\psi(+)$, $\psi(-)$ and ψ_0 are given in terms of

$$\dot{\psi}(+) = [\psi(+) , H]_P , \quad \dot{\psi}(-) = [\psi(-) , H]_P , \quad \dot{\psi}_0 = [\psi_0 , H]_P . \quad (3.18)$$

The variables (ψ, S) satisfy the Hamilton equations of motion:

$$\dot{\psi} = [\psi , H]_P , \quad \dot{S} = [S , H]_P . \quad (3.19)$$

In order to avoid unnecessary complication, we treat the following case:

$$\begin{aligned} S(+) = S(-) (= L) , \quad L = \frac{\hbar}{2} \Omega , \quad (\Omega(+) = \Omega(-) = \Omega) \\ S_0 = 0 . \end{aligned} \quad (3.20)$$

In this case, the relation (3.16) is reduced to

$$\hbar N = 4L , \quad \text{i.e.,} \quad N = 2\Omega . \quad (3.21)$$

The relation (3.21) shows us that we are interested in the closed shell system. Under the expectation values (3.12) and (3.13) and the condition (3.20), H can be reduced to

$$H = \epsilon \left(1 - \frac{\hbar}{S + \hbar} \right) \sqrt{(2L - S)(2L + S + 2\hbar)} \cos \psi - G(S^2 + \hbar S) . \quad (3.22)$$

The Hamiltonian (3.22) can be rewritten as

$$H = \epsilon \left(1 - \frac{\hbar}{\sigma} \right) \sqrt{(2\lambda)^2 - \sigma^2} \cos \psi - G\sigma(\sigma - \hbar) . \quad (3.23)$$

Here, λ and σ are defined by

$$\lambda = L + \hbar/2 , \quad \sigma = S + \hbar . \quad (3.24)$$

Concerning the parameters λ and σ , the following relation should be noted: Since there exists the restriction $|S(+) - S(-)| \leq S \leq S(+) + S(-)$ classically and quantum mechanically, for the case $S(+) = S(-)$, we have

$$\hbar \leq \sigma \leq 2\lambda . \quad (3.25)$$

Since (ψ, σ) is also in the canonical relation, we have the following Hamilton's equations:

$$\dot{\psi} = [\psi, H]_P = \frac{\partial H}{\partial \sigma} = \epsilon \left[\frac{\hbar}{\sigma^2} \sqrt{(2\lambda)^2 - \sigma^2} - \left(1 - \frac{\hbar}{\sigma} \right) \frac{\sigma}{\sqrt{(2\lambda)^2 - \sigma^2}} \right] \cos \psi - G(2\sigma - \hbar) , \quad (3.26a)$$

$$\dot{\sigma} = [\sigma, H]_P = -\frac{\partial H}{\partial \psi} = \epsilon \left(1 - \frac{\hbar}{\sigma} \right) \sqrt{(2\lambda)^2 - \sigma^2} \sin \psi . \quad (3.26b)$$

§4. Determination of the energy minimum point

One of our interests in this paper is to investigate how the energy minimum point induced by the mixed-mode coherent state $|c\rangle$ changes in the phase space (ψ, σ) as a function (ϵ, G) . The energy minimum point should be stationary in the phase space, and then, we have the condition

$$\dot{\psi} = 0 , \quad \dot{\sigma} = 0 . \quad (4.1)$$

The relation (3.26b) gives us

$$\sin \psi = 0 , \quad \text{i.e.,} \quad \cos \psi = -1 \text{ or } +1 . \quad (4.2)$$

It may be self-evident that the form (3.23) supports

$$\cos \psi = -1 . \quad (4.3)$$

The angle ψ does not depend on (ϵ, G) . Under the conditions (4.1) and (4.3), the relation (3.26a) is reduced, after some calculations, to

$$\begin{aligned} & \sigma^3 (\epsilon^2 - 4G^2((2\lambda)^2 - \sigma^2)) \\ &= \hbar \left(\epsilon(2\lambda)^2 - G\sigma^2 \sqrt{(2\lambda)^2 - \sigma^2} \right) \left(\epsilon + 2G\sqrt{(2\lambda)^2 - \sigma^2} \right) . \end{aligned} \quad (4.4)$$

By solving Eq.(4.4), we can determine the value of σ which makes the energy minimum. As a possible solution of the latter of Eq.(4.1), we obtain $\sigma = 2\lambda$, which is independent of ψ and ϵ . Substituting $\sigma = 2\lambda$ into the former of Eq.(4.1), we have $\cos \psi = -(4G\lambda/\epsilon)(4\lambda - \hbar)/(4\lambda - 2\hbar) \cdot \sqrt{\lambda\delta}$. Here, δ denotes an infinitesimal parameter defined as $\sigma = 2\lambda - \delta$. Then, under $\delta = 0$, $\cos \psi = 0$, i.e., $\psi = \pm\pi/2$, which is independent of ϵ and G . But, this solution cannot connect to the case $\sigma < 2\lambda$. Therefore, we do not adopt it. If we adopt the solution $\cos \psi = -1$, clearly, $4G\lambda/\epsilon$ becomes infinite under the condition $\delta \rightarrow 0$. This means that our two levels are degenerate.

Let us search the solution of Eq.(4.4). This relation can be simplified in the form

$$\rho^3 (1 - x^2(1 - \rho^2)) = \bar{k} \left(1 - \frac{x\rho^2}{2} \sqrt{1 - \rho^2} \right) (1 + x\sqrt{1 - \rho^2}) . \quad (4.5)$$

Here, \bar{k} , ρ and x are defined as

$$\bar{k} = \hbar/2\lambda , \quad (4.6)$$

$$\rho = \sigma/2\lambda , \quad x = 4G\lambda/\epsilon . \quad (4.7)$$

The relation (3.24) and the condition $L \geq 0$ give us

$$0 < \bar{k} \leq 1 . \quad (4.8)$$

Especially, we have

$$\begin{aligned} \bar{k} &\rightarrow 0 , \quad \text{if } \lambda \rightarrow \infty , \quad \text{i.e., } L \rightarrow \infty , \\ \bar{k} &= 1 , \quad \text{if } \lambda = \frac{\hbar}{2} , \quad \text{i.e., } L = 0 . \end{aligned} \quad (4.9)$$

If we take into account the correspondence to the original fermion system, the minimum value of Ω is equal to 1, and then, $L = \hbar/2$ and we have

$$0 < \bar{k} \leq 1/2 . \quad (4.10)$$

However, formally, we have the minimum value of $L = 0$. Further, the relation (3.25) gives us

$$\bar{k} \leq \rho \leq 1 . \quad (4.11)$$

Of course, we have

$$x \geq 0 . \quad (4.12)$$

First of all, on the basis of the relation (4.5), we treat the case \hbar (the Planck constant) $= 0$. In this case, we have a simple solution:

$$x = \begin{cases} \text{any value } (\geq 0) , & \text{for } \rho = 0 , \\ \frac{1}{\sqrt{1 - \rho^2}} , & \text{for } 0 < \rho < 1 . \quad (x \rightarrow \infty , \text{ if } \rho \rightarrow 1) \end{cases} \quad (4.13)$$

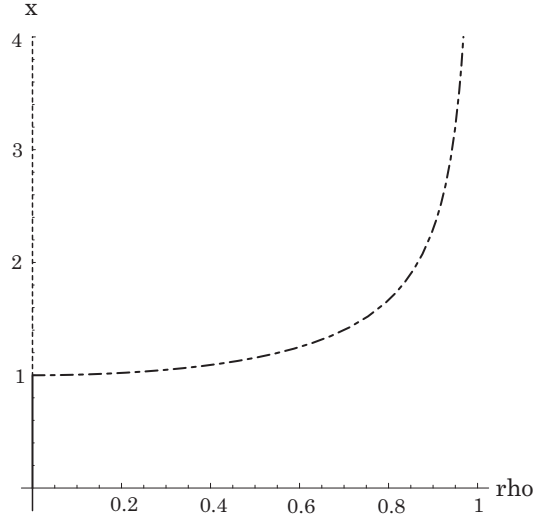


Fig. 1. The variable x is depicted as a function of ρ with $\hbar = 0$.

Inversely, the solution (4.13) are given as

$$\rho = \begin{cases} 0, & \text{for any value of } x(\geq 0), \\ \frac{\sqrt{x^2-1}}{x}, & \text{for } x \geq 1. \end{cases} \quad (4.14)$$

In Fig.1, we see a straight line consisting of solid and dotted lines along x -axis. This is one phase which is conventionally called as normal conducting phase. Further, we see that at the point $x = 1$ on the straight line (x -axis), a new phase is born toward the rectangular direction and it grows up to a branch, which is conventionally called as superconducting phase. In Fig.1, this phase is specified by a dot-dashed curve. The above is a conventionally accepted fact. In the region $x > 1$, there exist two phases and the energy of the superconducting phase is smaller than that of the normal conducting phase. From this reason, we chase the system under investigation along the solid to the broken line, the connection of which is not smooth. In this sense, conventionally we call this situation the sharp phase transition.

Our next task is to investigate the system under the condition $\hbar = \text{finite}$. For this purpose, it may be enough to solve the quadratic equation with respect to x , (4.5). In order to see the whole feature of the solution, for a moment, we treat the variables ρ and x in the region

$$-1 \leq \rho \leq 1, \quad -\infty < x < \infty. \quad (4.15)$$

Two solutions of Eq.(4.5) for the case $\bar{k} \neq 0$ and 1 are given exactly in simple forms:

$$x = \frac{1}{\sqrt{1-\rho^2}} \cdot \frac{\rho^3 - \bar{k}}{\rho^2 \left(\rho - \frac{\bar{k}}{2} \right)}, \quad (4.16a)$$

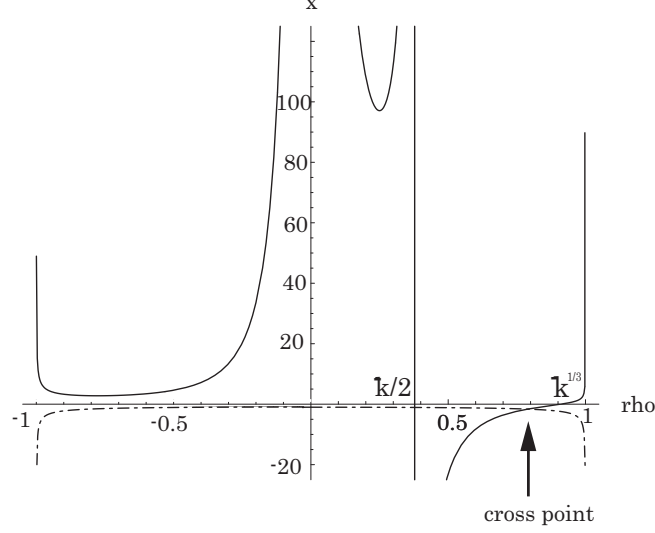


Fig. 2. The variable x is depicted as a function of ρ with $\hbar \neq 0$ ($k = 0.75$).

$$x = -\frac{1}{\sqrt{1-\rho^2}}. \quad (4.16b)$$

The value of ρ at the cross point of the solutions (4.16a) and (4.16b), which we denote ρ_P , is given by

$$\begin{aligned} \rho_P = \left(\frac{k}{2}\right)^{\frac{1}{3}} & \left[\left(1 + \frac{1}{2} \cdot \frac{\frac{1}{54} \left(\frac{k}{2}\right)^2}{1 + \sqrt{1 + \frac{1}{54} \left(\frac{k}{2}\right)^2}} \right)^{\frac{2}{3}} \right. \\ & \left. + \left(\frac{1}{2} \cdot \frac{\frac{1}{54} \left(\frac{k}{2}\right)^2}{1 + \sqrt{1 + \frac{1}{54} \left(\frac{k}{2}\right)^2}} \right)^{\frac{2}{3}} + \frac{1}{6} \left(\frac{k}{2}\right)^{\frac{2}{3}} \right]. \end{aligned} \quad (4.17)$$

Figure 2 shows the behavior of x as a function of ρ . The solid curves show the behaviors of x given in the relation (4.16a) in the region ($0 \leq \rho \leq 1$, $0 \leq x < \infty$) and in the other region, respectively. The dot-dashed curve shows the behavior of the solution (4.16b). We can see that between Figs.1 and 2, there exists quite big difference. In §6, we will discuss this difference. At first sight, Figs.1 and 2 tell us that, under the limit \hbar (k) $\rightarrow 0$, the solid curve in the region $k^{1/3} \leq \rho \leq 1$ of Fig.2 approaches to the solid line and the broken curve in Fig.1 and the solid curve in the region $0 \leq \rho \leq k^{1/3}$ is compressed and becomes the dotted line in Fig.1. Through the discussion in §6, the above view will be answered in the negative.

§5. Approximate solution for the relation determining the energy minimum point

As was shown in Eqs.(4.16a) and (4.16b), we can express x in terms of ρ . However, in order to understand the behavior of the system, for example, the change of the energy, in relation to the interaction strength, we must express ρ as a function of x . Numerically, it is possible and easy. But, analytical expression may be also helpful for the understanding of the system. In the present case, the exact form cannot be almost expected and we will try to the approximate form.

Let us investigate the case of the smooth curve in the region $\bar{k}/2 \leq \rho \leq 1$ in Fig.2. First, we note Eq.(4.16a). This equation can be rewritten in the form

$$x = \frac{\rho - \bar{k}^{1/3}}{(1 - \rho)(\rho - \bar{k}/2)} \cdot \sqrt{\frac{1 - \rho}{1 + \rho}} \cdot \frac{\rho^2 + \bar{k}^{1/3}\rho + \bar{k}^{2/3}}{\rho^2} . \quad (5.1)$$

Then, we define the quantity y in the form

$$y = \sqrt{\frac{1 + \rho}{1 - \rho}} \cdot \frac{\rho^2}{\rho^2 + \bar{k}^{1/3}\rho + \bar{k}^{2/3}} \cdot x . \quad (5.2)$$

With the quantity y , the relation (5.1) is formally reduced to the quadratic equation for ρ :

$$y = f(\rho) , \quad f(\rho) = \frac{\rho - \bar{k}^{1/3}}{(1 - \rho)(\rho - \bar{k}/2)} . \quad (5.3)$$

Formally, from the relation (5.3), we have

$$\rho = F(y) , \quad F(y) = \frac{\bar{k}y - 2\bar{k}^{1/3}}{\left(1 + \frac{\bar{k}}{2}\right)y - 1 - \sqrt{\left(1 - \frac{\bar{k}}{2}\right)^2 y^2 - 2\left(1 + \frac{\bar{k}}{2} - 2\bar{k}^{1/3}\right)y + 1}} . \quad (5.4)$$

If $x \rightarrow -\infty, 0$ and $+\infty$, i.e., $y \rightarrow -\infty, 0$ and $+\infty$, we have the relations $F(-\infty) = \bar{k}/2$, $\bar{k}^{1/3}$ and 1. Therefore, at the three characteristic points, we are able to obtain the exact results. From this reason, we do not adopt another branch for the solutions of the quadratic equation.

Next, we consider the case of the points except $x \rightarrow -\infty, 0$ and $+\infty$. First, we define a series $\{\rho^{(0)}, \rho^{(1)}, \rho^{(2)}, \dots\}$ obeying the following recursion relation:

$$\begin{aligned} \rho^{(k)} &= F(y^{(k-1)}) , & y^{(k-1)} &= f(\rho^{(k-1)}) , \\ k &= 1, 2, 3, \dots \end{aligned} \quad (5.5)$$

Here, we note that the function $f(y)$ is monotone-increasing. Then, if there exist the limiting value $\lim_{k \rightarrow \infty} \rho^{(k)} = \rho$, ρ is determined as a function of ρ by the condition

$$\rho = F(y) , \quad y = f(\rho) . \quad (5.6)$$

The relation (5.6) is nothing but the set of the relations (5.3) and (5.4). Under appropriate choice for the initial value $\rho^{(0)}$, iteratively, we obtain $\rho^{(k)}$ from $k = 1$ and after infinite iteration, we may obtain the exact solution which do not depend on $\rho^{(0)}$. If we stop the iteration half way, the result is an approximated one which depends on $\rho^{(0)}$. As a possible choice of the initial condition, we adopt the value of ρ satisfying the form (4.16a) at $x = 1$, which will be denoted as ρ_c :

$$1 = \frac{1}{\sqrt{1 - \rho_c^2}} \frac{\rho_c^3 - \bar{k}}{\rho_c^2(\rho_c - \bar{k}/2)} . \quad (5.7)$$

The relation (5.7) leads us to

$$\bar{k} = \frac{\rho_c^5}{(1 + \sqrt{1 - \rho_c^2}) \left(1 - \frac{\rho_c^2}{2} \sqrt{1 - \rho_c^2}\right)} . \quad (5.8)$$

Since the right-hand side of the relation (5.8) is increasing, and then, \bar{k} and ρ_c are in the relation of one-to-one correspondence. Therefore, instead of \bar{k} , we can use ρ_c defined in the relation (5.8). Therefore, the cases $x = -\infty, 0, 1$ and $+\infty$ do not depend on the iteration.

§6. Examination in the case $\bar{k} \rightarrow 0$ and 1 and its related phase transition

In §4, we showed the behavior of x with respect to ρ in the case where \bar{k} does not approach to 0 and 1. In this section, we discuss the case where \bar{k} approaches to 0 and 1. First, we treat the case $\bar{k} \rightarrow 0$. In the case $\rho < \rho_c$, we have

$$x = \frac{\rho^3 - \bar{k}}{\sqrt{1 - \rho^2} \rho^2 \left(\rho - \frac{\bar{k}}{2}\right)} \longrightarrow \frac{1}{\sqrt{1 - \rho^2}} , \quad (\bar{k} \rightarrow 0) . \quad (6.1)$$

Further, in the case $\rho = \rho_c$, we have

$$1 = \frac{\rho_c^3 - \bar{k}}{\sqrt{1 - \rho_c^2} \rho_c^2 \left(\rho_c - \frac{\bar{k}}{2}\right)} \longrightarrow \frac{1}{\sqrt{1 - \rho_c^2}} , \quad (\bar{k} \rightarrow 0) ,$$

namely,

$$\rho_c \longrightarrow 0 . \quad (\bar{k} \rightarrow 0) \quad (6.2)$$

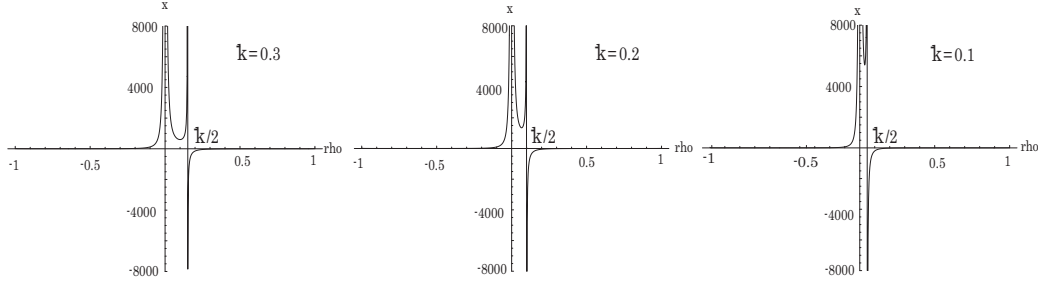


Fig. 3. The variable x is depicted as a function of ρ with $k \rightarrow 0$ ($k = 0.3, 0.2, 0.1$).

In the case where ρ is a little bit larger than $k^{1/3}$, we can put $\rho = (\alpha k)^{1/3}$ ($\alpha > 1$). Then, we have

$$\begin{aligned} x &= \frac{\alpha k - k}{\sqrt{1 - (\alpha k)^{2/3}} (\alpha k)^{2/3} \left((\alpha k)^{1/3} - \frac{k}{2} \right)} \\ &= \frac{\alpha - 1}{\sqrt{1 - (\alpha k)^{2/3}} \left(\alpha - \frac{1}{2} \alpha^{2/3} k^{2/3} \right)} \longrightarrow 1 - \frac{1}{\alpha}, \quad (k \rightarrow 0). \end{aligned} \quad (6.3a)$$

The above relation leads us to

$$\rho \longrightarrow 0, \quad x \longrightarrow 1 - \frac{1}{\alpha}, \quad \left(0 < 1 - \frac{1}{\alpha} < 1 \right), \quad (k \rightarrow 0). \quad (6.3b)$$

The relation (6.3b) tells that under the limit $k \rightarrow 0$, $\rho \rightarrow 0$ and x takes arbitrary value between 0 and 1. From the above three cases, we see that x in the region $k^{1/3} < \rho < 1$ in Fig.2 is reduced to the solid line and the dot-dashed curve in Fig.1. Next, we treat the case where ρ is a little bit smaller than $k^{1/3}$. In this case, we can set up $\rho = (k/\alpha)^{1/3}$ ($\alpha > 1$). Then, in the same process, we have

$$x \longrightarrow 1 - \alpha, \quad (k \rightarrow 0). \quad (6.4a)$$

The above means the following situation:

$$\rho \longrightarrow 0, \quad x \longrightarrow 1 - \alpha, \quad (1 - \alpha < 0). \quad (k \rightarrow 0) \quad (6.4b)$$

The above situation can be interpreted as that under $k \rightarrow 0$, $\rho \rightarrow 0$ and x takes negative arbitrary value and in Fig.1, the straight line between 0 and 1 continues to the negative region of x . In the case where ρ is in the region $0 < \rho < k/2$, we can put $\rho = \alpha(k/2)$, ($0 < \alpha < 1$). Then, we have

$$\begin{aligned} x &= \frac{(\alpha k/2)^3 - k}{\sqrt{1 - (\alpha k/2)^2} (\alpha k/2)^2 (\alpha k/2 - k/2)} \\ &= \frac{8/k^2 - \alpha^3}{\sqrt{1 - \alpha^2 k^2/4} \alpha (1 - \alpha)} \longrightarrow +\infty. \quad (k \rightarrow 0) \end{aligned} \quad (6.5)$$

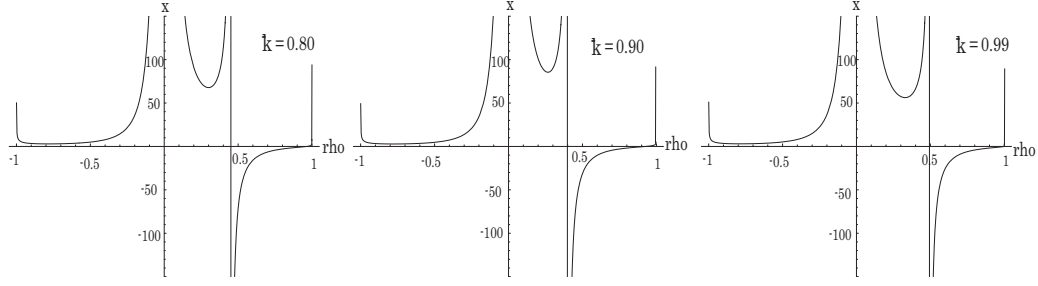


Fig. 4. The variable x is depicted as a function of ρ with $k \rightarrow 1$ ($k = 0.8, 0.9, 0.99$).

This situation is very interesting. The solid curve in the region $0 < \rho < k/2$ in Fig.2 disappears to $+\infty$. Therefore, the dotted line in Fig.1 does not correspond to this curve. Next, in the case where ρ is a little bit smaller than 0, we can set up $\rho = -(\alpha k)^{1/3}$ ($\alpha > 0$). Then, x can be expressed in the form

$$\begin{aligned} x &= \frac{\alpha k + k}{\sqrt{1 - (\alpha k)^{2/3}} \alpha^{2/3} k^{2/3} \left(\alpha^{1/3} k^{1/3} + \frac{k}{2} \right)} \\ &= \frac{1}{\sqrt{1 - (\alpha k)^{2/3}}} \cdot \frac{\alpha + 1}{\alpha + \frac{1}{2} \alpha^{2/3} k^{2/3}} \longrightarrow 1 + \frac{1}{\alpha}, \quad (k \rightarrow 0). \end{aligned} \quad (6.6a)$$

The above means

$$\rho \longrightarrow 0, \quad x \longrightarrow 1 + \frac{1}{\alpha}, \quad \left(1 + \frac{1}{\alpha} > 1 \right). \quad (k \rightarrow 0) \quad (6.6b)$$

In this case, under the limit $k \rightarrow 0$, $\rho \rightarrow 0$ and x is positive arbitrary and this situation corresponds to the dotted line starting at $x = 1$ in Fig.1. However, this situation comes from the negative region of ρ , which is physically unacceptable. Further, in the case $\rho \ll -(\alpha k)^{1/3}$, $x \rightarrow 1/\sqrt{1 - \rho^2}$ ($k \rightarrow 0$).

Next, we treat the case $k \rightarrow 1$. In this case, we divide two situations. One is the case $k^{1/3} < \rho$. If we put $k = 1 - \delta k$ and $\rho = 1 - \delta \rho$, we have $\delta \rho < \delta k/3$. Then, we put $\delta \rho = \alpha \delta k/3$ ($0 < \alpha < 1$). Under the condition that δk ($\delta \rho$) is sufficiently small, x can be expressed as

$$x = \frac{(1 - \alpha)\delta k}{\frac{2}{3}\alpha\delta k \times \frac{1}{2}} \longrightarrow 3 \left(\frac{1}{\alpha} - 1 \right), \quad \left(\frac{1}{\alpha} - 1 > 0 \right). \quad (k \rightarrow 0) \quad (6.7)$$

In this case, under the limit $k \rightarrow 1$, $\rho \rightarrow 1$ and x is positive arbitrary. The above means that the solid curve between $\rho = k^{1/3}$ and 1 becomes the straight line on $\rho = 1$. On the other hand, for the region $k/2 \leq \rho \leq k^{1/3}$, the limit $k \rightarrow 1$ gives

$$x \longrightarrow -\sqrt{\frac{1 - \rho}{1 + \rho}} \frac{1 + \rho + \rho^2}{\rho^2(\rho - 1/2)}. \quad (6.8)$$

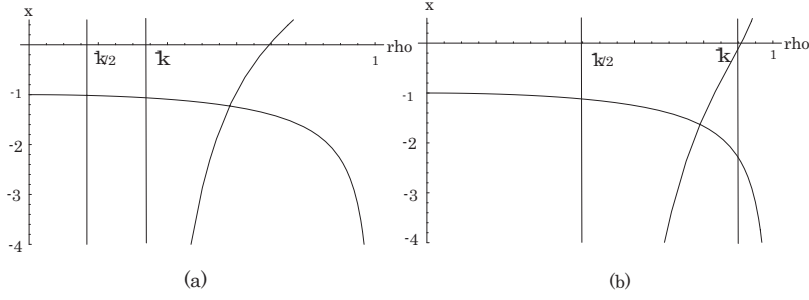


Fig. 5. The cross points are shown in the case (a) $\bar{k} = 0.33$ and (b) $\bar{k} = 0.90$.

The above shows that the solid curve between $\rho = \bar{k}/2$ and $\bar{k}^{1/3}$ in Fig.2 becomes the curve showing by the relation (6.8). The other cases do not have characteristic change. However, as was already mentioned, $\bar{k} \rightarrow 1$ corresponds to $L \rightarrow 0$. In the system in which the seniority is zero, L starts from $L = 1/2$, and then, the present case is not so interesting. But, in the case where the seniority is not zero, it may be interesting.

On the basis of the above-shown results, let us consider the phase transition in our model. In the present case, physically interesting feature can be seen in the solid curve between $\rho = \bar{k}^{1/3}$ and 1 in Fig.2. We cannot find any non-smoothness or discontinuity. Of course, under the limit $\bar{k} \rightarrow 0$, the result appears as the phase transition which is conventionally accepted. Therefore, in the finite \hbar and λ , we should not expect the phase transition. Further, we cannot find any curve which leads us to the dotted line in Fig.1 from the side of the physically acceptable region. Therefore, the dotted line in Fig.1 should be negative understood. From the above examination, we can learn that the limiting process $\hbar \rightarrow 0$ must be carefully performed. All do not follow this process.

Let our model accepted in the region $x < 0$ (the repulsive pairing interaction). In Fig.2, we see that there exists a cross point of two different curves. Noting the relation (4.11) ($\bar{k} \leq \rho \leq 1$), we examine characteristic feature of the crossing. There exist two types of the crossing which show in Fig.5(a) and 5(b). Both figures are characterized by $\rho_P > \bar{k}$ and $\bar{k}/2 < \rho_P < \bar{k}$, respectively. Here, ρ_P is shown in the relation (4.17). From Fig.5, we learn that there exist three types of non-smoothness and discontinuity, which are illustrated by

Fig. 6. The three types of non-smoothness and discontinuity for the crossing point ρ_P are illustrated in the case $\bar{k} < \rho_P$.

Fig. 7. A type of non-smoothness and discontinuity for the crossing point ρ_P is illustrated in the case $\bar{k} > \rho_P$.

Fig.6. Following the change of the interaction strength, the system moves along the solid curve. At the present, we do not judge which should be chose in the three cases. In the case of Fig.5(b), we can draw the picture shown in Fig.7. Discontinuity appears in the interaction strength. Thus, we can conclude that in our present model, the phase transition appears in the case of the negative interaction strength. However, at the present stage, we cannot judge if the above phase transition has its realistic meaning.

§7. Discussion

Finally, in this section, we discuss some characteristic points of the energy E . The energy E is obtained by substituting the solution of Eq.(4.1) into H shown in the relation (3.22):

$$E = -\epsilon \cdot 2\lambda \left[\left(1 - \frac{k}{\rho}\right) \sqrt{1 - \rho^2} + \frac{x}{2}\rho(\rho - k) \right]. \quad (7.1)$$

In Fig.8, we show numerical results. In the case $x \rightarrow \infty$ (two levels are degenerate), we have $\rho = 1$ and E is reduced to

$$E_1 = -G \cdot 2L(2L + \hbar) = -\hbar^2 G \Omega(\Omega + 1). \quad (7.2)$$

The form (7.2) is the same as that shown exactly. On the other hand, in the case $x = 0$ (free fermions), we have $\rho = k^{1/3}$ and E is written as

$$\begin{aligned} E_0 &= -\epsilon \cdot 2\lambda \left(1 - k^{\frac{2}{3}}\right)^{\frac{3}{2}} \\ &= -\hbar\epsilon \left(\frac{2L + \hbar}{\hbar}\right) \left(1 - \left(\frac{\hbar}{2L + \hbar}\right)^{\frac{2}{3}}\right)^{\frac{3}{2}} \\ &= -\frac{\hbar\epsilon}{2} \cdot 2 \left[(\Omega + 1)^{\frac{2}{3}} - 1\right]^{\frac{3}{2}}. \end{aligned} \quad (7.3)$$

If Ω is large, the leading term of E is given by

$$E \approx -\frac{\hbar\epsilon}{2} \cdot 2\Omega \left(= -\frac{\hbar\epsilon}{2}N\right). \quad (7.4)$$

The above form tells us that the lower level is fully occupied. Therefore, we can conclude that in the case of large Ω , our result is reduced to the exact one and for the problem of $k \rightarrow 0$, it may be favorable.

Fig. 8. The ground state energy normalized by ϵ is depicted as a function of the interaction strength $x = 4G\lambda/\epsilon$ with $k = 1/3$ (solid curve) together with E_0 and E_1 .

§8. Summary

In this paper, we analyzed the aspect of phase change in the two-level pairing model governed by the $su(2) \otimes su(2)$ -algebra. This model can be described in terms of four kinds of boson operators by means of the Schwinger boson representation of the $su(2)$ -algebras. We used the variational approach with the mixed-mode coherent state introduced in Ref.5). This state is constructed, as is similar to the usual coherent state, by using the raising operators of the $su(2)$ - and the $su(1,1)$ -algebras whose generators are defined by the Schwinger bosons used in this model.

In the classical limit $\hbar \rightarrow 0$, in which the Planck constant is taken to be 0 at the first time, if the force strength is smaller than a certain critical value, the trivial phase, which is called the non-condensed phase, is realized. At the critical point with respect to the force strength, a new branch appears and this branch reveals the existence of the so-called condensed phase. However, the non-condensed phase still exists as a result of the trivial solution if the force strength is larger than the critical value. This situation is well known in the usual Hartree-Bogoliubov approximation and the sharp phase transition is realized in this treatment.

However, this behavior is not true because the sharp phase transition is not realized in this many-body system and this fact can easily be verified in this exactly solvable algebraic model. Figures 2 shows the aspect of phase structure. Furthermore, in Fig.3, the aspect of the phase change in the semi-classical limit is shown clearly. It is seen in Fig.2 that the two branches exist in the physical region in $0 < \rho \leq 1$, where 0 is included when $\hbar \rightarrow 0$. If \hbar approaches to 0, one branch, which is located in $0 < \rho < k/2$, goes to infinity of x and this branch disappears in the limit $\hbar \rightarrow 0$. Another branch, which is located in $k/2 < \rho \leq 1$ and $x \geq 0$, is reduced to the physical branch in Fig.1, which is represented by solid and dash-dotted curves. However, the branch of “the non-condensed phase” in the classical limit, which is represented by the dotted line in Fig.1 in $x > 1$, appears from the unphysical region in $\rho < 0$ in Fig.2. So, this solution is not a physical solution originally. Thus, this “non-condensed phase” is not realized in the physical system. Therefore, the sharp phase transition does not occur if \hbar is taken as finite value.

In our model, there exist a cross point of two branches represented by two curves in Fig.2 in the negative region of x . Because the variable x is proportional to the force strength G , this cross point is realized in the case of the “repulsive pairing”. Thus, the sharp phase transition may be realized in the case of the negative interaction strength.

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Appendix A

—— A method for solving the relation (5.5) ——

Let an infinite series $\{y^{(0)}, y^{(1)}, y^{(2)}, \dots\}$ obey the following condition characterized by a function $f(x, y)$:

$$y^{(k)} = (f(x, y^{(k-1)}))^{\nu} . \quad (k = 1, 2, 3, \dots) \quad (\text{A}\cdot 1)$$

Here, ν denotes real parameter and x plays a role of real variable ($0 \leq x < \infty$). Further, we regard $f(x, y)$ as a monotone-increasing or -decreasing function. If there exists the limiting value $\lim_{k \rightarrow \infty} y^{(k)} (= y)$, y is determined as a function of x by the condition

$$y = (f(x, y))^{\nu} . \quad (\text{A}\cdot 2)$$

The above means that the variable y governed by the relation (A.2) is expressed as a function of x by calculating the limiting value $\lim_{k \rightarrow \infty} y^{(k)}$ of the series $\{y^{(0)}, y^{(1)}, y^{(2)}, \dots\}$ obeying the condition (A.1). On the basis of the above idea, in this Appendix, we develop a systematic method for obtaining an approximate expression of y as a function of x . Of course, x and y obey the relation (A.2), and hereafter, we treat the following case: Through the relation (A.2), we are able to obtain the values of y at $x = 0$ and at $x \rightarrow \infty$. These two values are denoted as y_0 and y_{∞} , respectively:

$$y_0 = (f(0, y_0))^{\nu} , \quad y_{\infty} = (f(\infty, y_{\infty}))^{\nu} . \quad (\text{A}\cdot 3)$$

Hereafter, we denote any quantity X at $x = 0$ and $x \rightarrow \infty$ as X_0 and X_{∞} , respectively. The derivative $(dy/dx)_0$ is given as

$$\left(\frac{dy}{dx}\right)_0 = \frac{\left(\frac{\partial f}{\partial x}\right)_0}{\frac{1}{\nu y_0^{\nu-1}} - \left(\frac{\partial f}{\partial y}\right)_0} . \quad (\text{A}\cdot 4)$$

The quantities $(\partial f/\partial x)_0$ and $(\partial f/\partial y)_0$ are obtained by putting $x = 0$ and $y = y_0$ for $\partial f/\partial x$ and $\partial f/\partial y$ as functions of x and y , respectively.

Let us set up the following form for $y^{(0)}$:

$$y^{(0)} = \left(\frac{y_\infty^\frac{1}{\nu} x + y_0^\frac{1}{\nu} \alpha}{x + \alpha} \right)^\nu . \quad (\text{A}\cdot 5)$$

Independently from α , we have

$$y_0^{(0)} = y_0 , \quad y_\infty^{(0)} = y_\infty . \quad (\text{A}\cdot 6)$$

We choose α so as to make $y^{(0)}$ satisfy the relation (A·4):

$$\alpha = \nu y_0 \left[\left(\frac{y_\infty}{y_0} \right)^\frac{1}{\nu} - 1 \right] \frac{\frac{1}{\nu y_0^{\nu-1}} - \left(\frac{\partial f}{\partial y} \right)_0}{\left(\frac{\partial f}{\partial x} \right)_0} . \quad (\text{A}\cdot 7)$$

Therefore, naturally, we have

$$\left(\frac{dy^{(0)}}{dx} \right)_0 = \left(\frac{dy}{dx} \right)_0 . \quad (\text{A}\cdot 8)$$

Through the relation (A·1), we obtain $y^{(1)}$ from $y^{(0)}$ given by the form (A·5). By performing the above process successively, we can calculate $y^{(k)}$ for any k . If k is larger, the accuracy of the approximation may be better. We can prove the following form:

$$\begin{aligned} y_0^{(k)} = y_0^{(0)} = y_0 , \quad y_\infty^{(k)} = y_\infty^{(0)} = y_\infty , \\ \left(\frac{dy^{(k)}}{dx} \right)_0 = \left(\frac{dy^{(0)}}{dx} \right)_0 = \left(\frac{dy}{dx} \right)_0 . \quad (k = 2, 3, 4, \dots) \end{aligned} \quad (\text{A}\cdot 9)$$

Therefore, we can conclude that, for any k , the behaviors of $y^{(k)}$ near $x = 0$ and $x \rightarrow \infty$ do not change from those for $k = 0$.

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